MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

II. SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let the radii of gyration of the shell and sphere, k', k'', be given by

$$k'^2 = \frac{2}{5}\frac{R^5 - r^5}{R^3 - r^3},$$

$$k''^2 = \frac{2}{5}r_1^2;$$

C_0T_0 the radius of the shell to the tangent point on the inclined plane initially; E_0 the common point initially of the inner surface of the shell and sphere; CT_0 the position of C_0T_0 after any time from the beginning of motion of the system; D the center of the sphere at the same time t; CT the radius of the shell to the tangent point T of shell and inclined plane at the same instant; φ = angle TCT_0; DE = the radius of the sphere to the tangent point of the inner surface of the shell and the sphere; θ = the angle DE makes with the vertical through D;

$$s = TT_0 = CC_0 = R\phi;$$

CH = a perpendicular from C upon C_0E_0 cutting the latter at H;

$$r - r_1 = r';$$

C_0 the coördinate origin; C_0C the x-axis; C_0T_0 the y-axis; x, y the coördinates of D; then $\angle C_0CH = \alpha$; $\angle C_0CD = \pi/2 - \theta + \alpha$; and then, F being the foot of the perpendicular from D upon CC_0,

$$x = s - CF = R\varphi - r'\cos DCF = R\varphi - r'\sin(\theta - \alpha);$$

$$y = DF = r'\sin DCF = r'\cos(\theta - \alpha).$$

The dynamic conditions for the motion of the sphere can be most clearly indicated by noticing that the initial point E_0 remains in contact with the inner surface of the shell, while the sphere has an angular velocity $\dot\theta$, $\dot\varphi$ being that of the shell.

Let M, m, be the masses of the shell and of the sphere; T, V, the kinetic energy, and potential energy; then the kinetic potential equation for the system is

$$T = \tfrac{1}{2}M(k'^2\dot\varphi^2 + \dot s^2) + \tfrac{1}{2}m(\dot x^2 + \dot y^2 + k''^2\dot\theta^2)$$

$$= Mgs\sin\alpha + mg(r'\cos\theta + s\sin\alpha) + C = V.$$

But

$$\dot x = R\dot\varphi - r'\cos(\theta - \alpha)\dot\theta,$$

$$\dot y = -r'\sin(\theta - \alpha)\dot\theta,$$

and

$$\dot s = R\dot\varphi,$$

whence, on substitution and reduction,

$$T = \tfrac{1}{2}\{M(k'^2 + R^2) + mR^2\}\dot\varphi^2 + \tfrac{1}{2}m\{-2Rr'\cos(\theta - \alpha)\dot\theta\dot\varphi + (r'^2 + k''^2)\dot\theta^2\}$$

$$= g\{(M + m)R\varphi\sin\alpha + mr'\cos\theta\} + C = V.$$

The Lagrangian equations

$$\frac{d}{dt}\frac{dT}{d\dot\varphi} - \frac{dT}{d\varphi} = \frac{dV}{d\varphi},$$

$$\frac{d}{dt}\frac{dT}{d\dot\theta} - \frac{dT}{d\dot\theta} = \frac{dV}{d\theta},$$

applied to the last result, give after simplifications,

$$\{(M+m)R^2 + Mk'^2\}\ddot{\varphi} - mRr' \cos(\theta - \alpha)\ddot{\theta} + mRr' \sin(\theta - \alpha)\dot{\theta}^2 = g(M+m)R \sin \alpha,$$

$$Rr' \cos(\theta - \alpha)\ddot{\varphi} - (r'^2 + k''^2)\ddot{\theta} = gr' \sin\theta.$$

Eliminating $\ddot{\varphi}$,

$$[(r'^2 + k''^2)\{(M+m)R^2 + Mk'^2\} + mR^2r'^2 \cos^2(\theta - \alpha)]\ddot{\theta} - mR^2r'^2 \sin(\theta - \alpha)\cos(\theta - \alpha)\dot{\theta}^2$$
$$= -gr'[\{(M+m)R^2 + Mk'^2\}\sin\theta + (M+m)R^2 \sin\alpha \cos(\theta - \alpha)].$$

Multiplying by $2\dot{\theta}$ and integrating

$$[(r'^2 + k''^2)\{(M+m)R^2 + Mk'^2\} + mr'^2R^2 \cos^2(\theta - \alpha)]\dot{\theta}^2$$
$$= gr'[2\{(M+m)R^2 + Mk'^2\}\cos\theta - 2(M+m)R^2 \sin\alpha \sin(\theta - \alpha)] + C',$$

which is of the same general form as (7), p. 351, this MONTHLY for November, 1916.

NUMBER THEORY.

235. Proposed by W. D. CAIRNS, Oberlin College.

Prove that $n = 1$ is the only positive integer for which $n^4 + 4$ is a prime.

SOLUTION BY WM. E. PATTEN, Government Institute of Technology, Shanghai, China.

$$n^4 + 4 = (n^4 + 4n^2 + 4) - 4n^2 = (n^2 + 2)^2 - (2n)^2 = (n^2 + 2n + 2)(n^2 - 2n + 2).$$

Therefore, $n^4 + 4$ is a prime, if at all, only for those values of n which make either $n^2 + 2n + 2 = 1$, or $n^2 - 2n + 2 = 1$, since each of the factors of $n^4 + 4$ given above is integral in value when n is integral, and both are positive when n is positive.

(1) When $n^2 + 2n + 2 = 1$, then $n = -1$.

(2) When $n^2 - 2n + 2 = 1$, then $n = +1$. When $n = +1$, then $n^4 + 4 = 5$, a prime.

Therefore, $n^4 + 4$ is a prime for $n = 1$, and for no other positive integral values of n.

Also solved by ELMER SCHUYLER, FRANK IRWIN, HORACE OLSON, ELIJAH SWIFT, H. H. CLARK, ELIZABETH B. DAVIS, NORMAN ANNING, L. G. WELD, and the PROPOSER.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy + z = \square, \quad yz + x = \square, \quad \text{and} \quad xz + y = \square.$$

SOLUTION BY ARTEMAS MARTIN, LL.D., Washington, D. C.

Assume $x = n^2$, $y = (n+1)^2$; then the given equation becomes

$$n^2(n+1)^2 + z = \square, \quad (n+1)^2 z + n^2 = \square, \quad n^2 z + (n+1)^2 = \square.$$

Put

$$n^2(n+1)^2 + z = a^2.$$

Assume $a = n^2 + n + b$, and the last equation becomes

$$n^2(n+1)^2 + z = a^2 = (n^2 + n + b)^2,$$

from which we immediately find

$$z = b(2n^2 + 2n + b).$$

Substituting in

$$(n+1)^2 z + n^2 = \square,$$

we have

$$b(n+1)^2(2n^2 + 2n + b) + n^2 = \square = c^2,$$